CSE 105 Discussion – Week 4

PUMPING LEMMA AND PDA

Do non-regular languages exist?

Yes! Why?

Q1: What does it mean for a language to be regular? (might have multiple right answers)

A. Finite
B. Can be recognized by an NFA/DFA
C. Can be described by a Regex

Q2: What is the cardinality of the set of all Regex over some alphabet?

A. Finite
B. Countable
C. Uncountable

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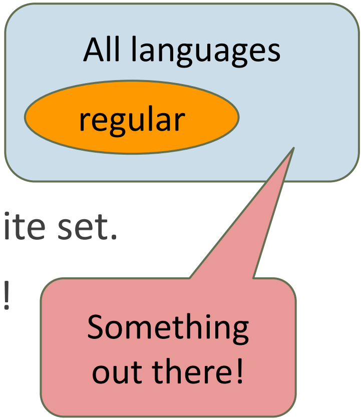

Now, the set of all languages is uncountable since it is the powerset of an infinite set.

Also note that the set of languages NFA/DFA/Regex can describe are the same!

Intuitions about non-regular langs

- 0*1*
- {1^n 0^m | n, m > 1}
- {0, 00, 0000, 00000000, …}
- {0, 000, 00000, …}
- {0^n 1^m | n > m > 0}

Intuitions about non-regular langs

- 0*1* regular
- {1^n 0^m | n, m > 1} regular
- {0, 00, 0000, 00000000, …} non-regular
- {0, 000, 00000, …} regular
- {0^n 1^m | n > m > 0} non-regular

Pumping Lemma

If A is a regular language then there is a number p (the pumping length) where if s is any string in A of length at least p then s may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$, and // The loop is nonempty
- For each $i \geq 0, xy^i z \in A$ // Pumping the loop any # of times creates other strings in A
- $|xy| \leq p$ // The loop appears in the first p characters

For regular languages we can set $p >$ # of states in DFA recognizing language

For finite languages we can set $p >$ length of longest string in language

Pumping Lemma T/F

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Are the following statements True or False?

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- L is regular $\rightarrow$ L has a pumping length **True**, this is what the Pumping Lemma tells us
- L is not regular $\rightarrow$ L does not have a pumping length

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- L has a pumping length $\rightarrow$ L is regular

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- L is not regular $\rightarrow$ L does not have a pumping length *False*, this is the converse of Pumping Lemma
- L has a pumping length $\rightarrow$ L is regular *False*, this is the inverse of Pumping Lemma
- L does not have a pumping length $\rightarrow$ L is not regular

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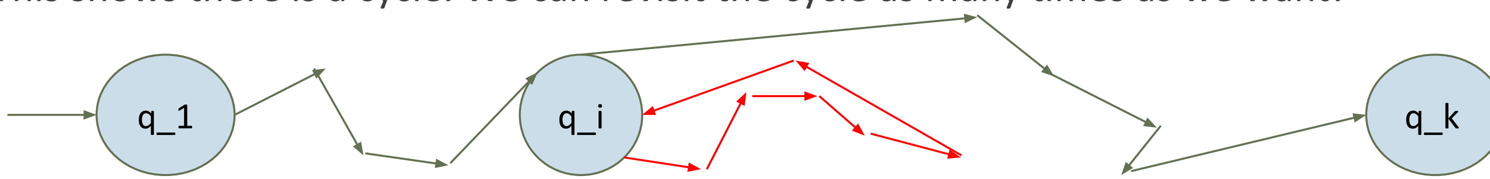
- L is not regular $\rightarrow$ L does not have a pumping length *False*, this is the converse of Pumping Lemma
- L has a pumping length $\rightarrow$ L is regular *False*, this is the inverse of Pumping Lemma
- L does not have a pumping length $\rightarrow$ L is not regular **True**, this is the contrapositive of the Pumping Lemma
  - This is the statement we use to prove a language is not regular

Proof Sketch

- Suppose a language is regular, then it must have a DFA that recognizes it.
- DFA has finite amount of states, let's say k.
- Let s be a string of length $n \geq k$.
- Suppose s is accepted, that means after n transitions, we land in an accept state.
- Though the journey to accept state, we've visited n+1 states including the start.
- Now, $n+1 > k$, so at least one state has been visited twice.
- Let's say the we visited q_1, q_2 q_i, … q_i, … with q_i visited at least two times.
- This shows there is a cycle. We can revisit the cycle as many times as we want!



Pumping Lemma – Formal Logic

If A is a regular language then there is a number p (the pumping length) where if s is any string in A of length at least p then s may be divided into three pieces, $s = xyz$ such that

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- $|xy| \leq p$ // The loop appears in the first p characters
- If A is a regular language then…
  - $\exists p\ (\forall s \in A\ |s| \geq p \rightarrow \exists x, y, z\ \left(s = xyz \wedge |y| > 0 \wedge |xy| \leq p \wedge \left(\forall i \in N\ xy^iz \in A\right)\right))$
- Contrapositive: Negate both sides and swap them
- If $\forall p(\exists s \in A\ |s| \geq p \wedge \forall x, y, z\ \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \left(\exists i \in N\ xy^iz \notin A\right)\right))$
  - …then A is a nonregular language
  - If we can show that all values of p are not pumping lengths for A then we have shown that A is nonregular

Pumping Lemma – Strategy

In proofs of nonregularity of language A using the pumping lemma our goal is to show

$$\forall p(\exists s \in A\ |s| \geq p \wedge \forall x, y, z\ \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \left(\exists i \in N\ xy^i z \notin A\right)\right))$$

- For any p

- There is a string s such that

- For any viable split of the string into x, y, and z

- We can choose some # of repetitions of y to get a string not in the language

Pumping Lemma – Strategy

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$$\forall p(\exists s \in A \ |s| \geq p \wedge \forall x, y, z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \left(\exists i \in N \ xy^i z \notin A\right)\right))$$

- For any p
  - Consider arbitrary p
- There is a string s such that
  - Choose a string s in terms of p (creative part)
- For any viable split of the string into x, y, and z
  - Define x, y, z according to PL conditions $|y| > 0 \wedge |xy| \leq p$
- We can choose some # of repetitions of y to get a string not in the language
  - Choose i such that $xy^i z$ is not in the language (other creative part)

Pumping Lemma – Example

- Consider the language $PAL = \{w \in \{0,1\}^* | w = w^R\}$, i.e. the set of all palindromes over $\{0,1\}$
- Show that PAL is nonregular using the pumping lemma
- WTS

$$\forall p(\exists s \in PAL\ |s| \geq p \wedge \forall x, y, z\left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \left(\exists i \in N\ xy^i z \notin PAL\right)\right))$$

Consider arbitrary pumping length p. WTS there is a valid string in PAL that can't be pumped.

Which string should we choose?

A. 111000111

B. 10^p1

C. 0^p10^p

D. 0^p1^p

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$$\forall p(\exists s \in PAL\ |s| \geq p \wedge \forall x, y, z\ \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \left(\exists i \in N\ xy^i z \notin PAL\right)\right))$$

Consider arbitrary pumping length p. WTS there is a valid string in PAL that can't be pumped.

Consider string $s = 0^p 1 0^p \in PAL$, where $|s| > p$, as desired.

Let $s = xyz$ where $x = 0^k, y = 0^j, z = 0^l 1 0^p$ such that $j > 0,$ and $k + j + l = p$.

WTS there is a value i such that $xy^i z \notin PAL$

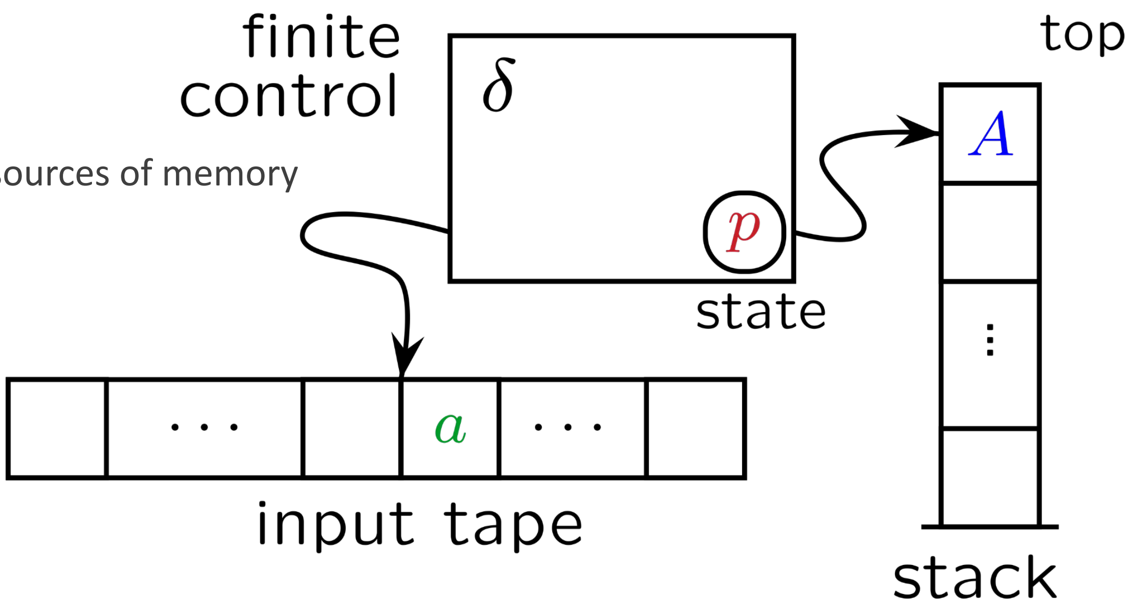
Consider $i = 0$. Then $xy^i z = xz = 0^k 0^l 1 0^p$. Since j > 0 then $k + l < k + j + l$.

Then $k + l < p$, so $0^k 0^l 1 0^p$ has an unequal number of leading and ending 0s, and therefore is not palindromic.

Therefore, $xy^0 z \notin PAL$ and p is not a pumping length for PAL. Thus PAL has no pumping length and is nonregular.

Pushdown Automata (PDA)

- What is the source of memory of an NFA?
  - The state it is in
  - That's it
- Now add stack
  - we now have two sources of memory



PDA Formal Description

DEFINITION 2.13

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta \colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

Compare And Contrast

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Non-deterministic!

$$\delta_{PDA}(state, char, pop) = \{(new\_state, push), \ldots\}$$

DEFINITION 1.37

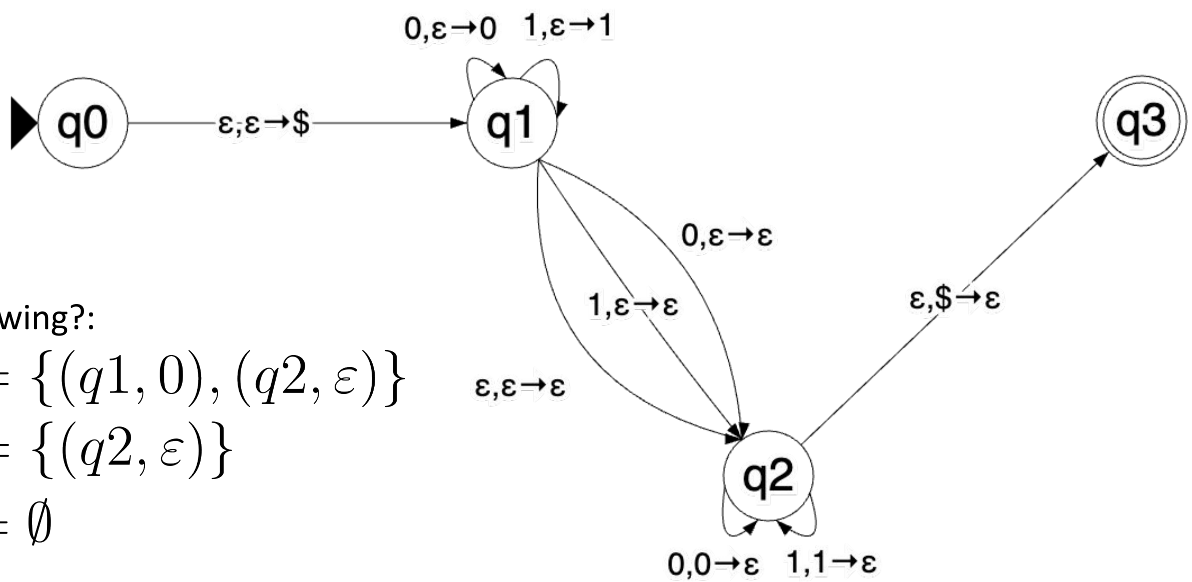
A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

$$\delta_{NFA}(state, character) = \{new\_state, \ldots\}$$

PDA Transition Practice

flapjs link



What are the following?:

$$\delta(q1, 0, \varepsilon) = \{(q1, 0), (q2, \varepsilon)\}$$

$$\delta(q2, 1, 1) = \{(q2, \varepsilon)\}$$

$$\delta(q_2, 0, \epsilon) = \emptyset$$

Convert Languages to PDA

$$\{a_1b_1a_2b_2\ldots a_nb_n \mid count(a_1a_2\ldots a_n) = count(b_1b_2\ldots b_n) \;\wedge\; b_1b_2\ldots b_n \in L(0^*1^*)\}$$

count(s) = number of 1s in s

- Need to keep track of 1s in even positions and make sure they match the number of 1s in odd positions
- All 1s in odd positions need to come after 0s
- Empty string is allowed

Sample machine

Convert Languages to PDA

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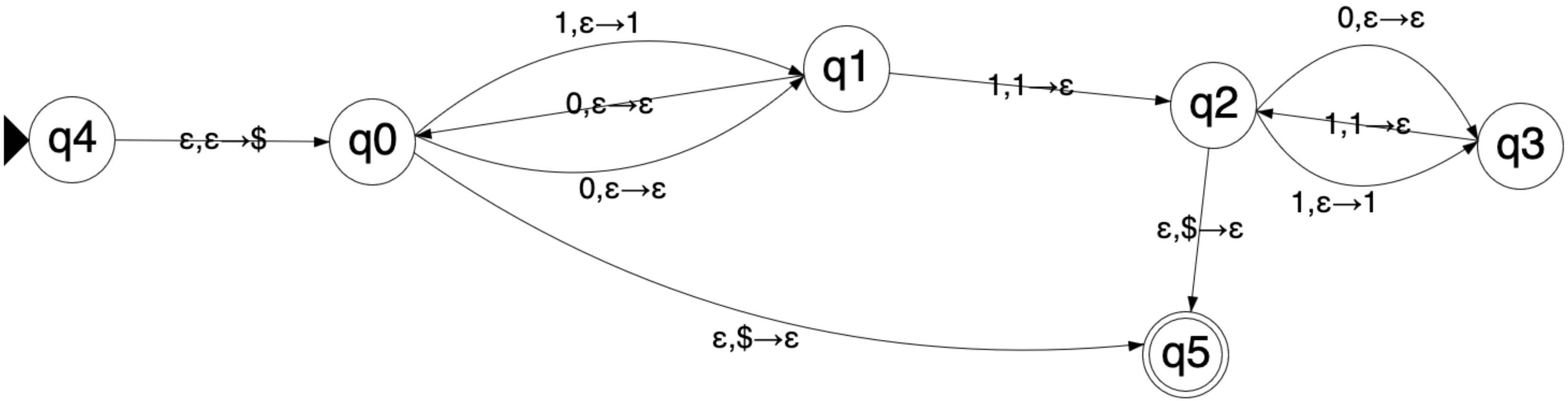
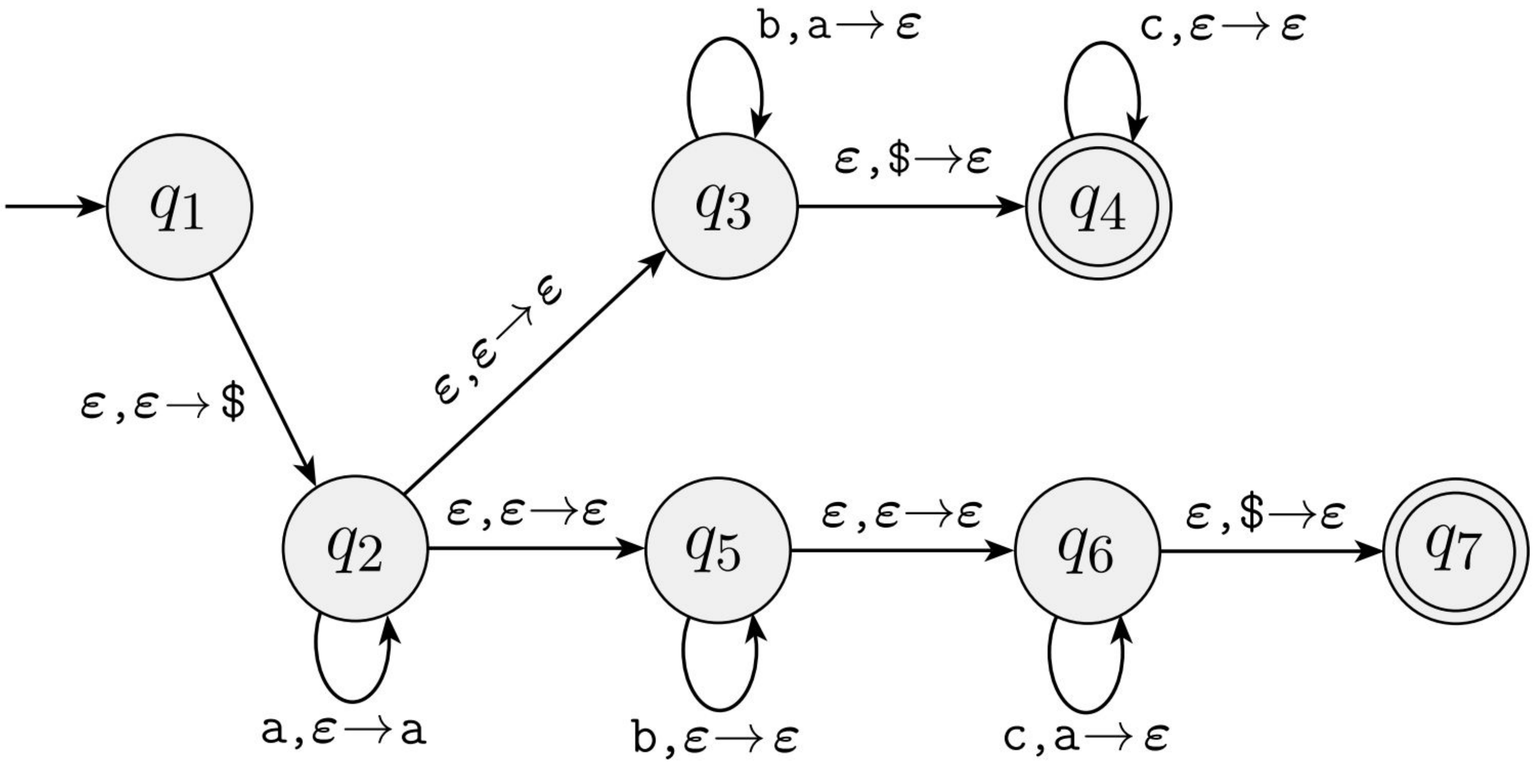


[flap.js link]

Decipher PDA Language



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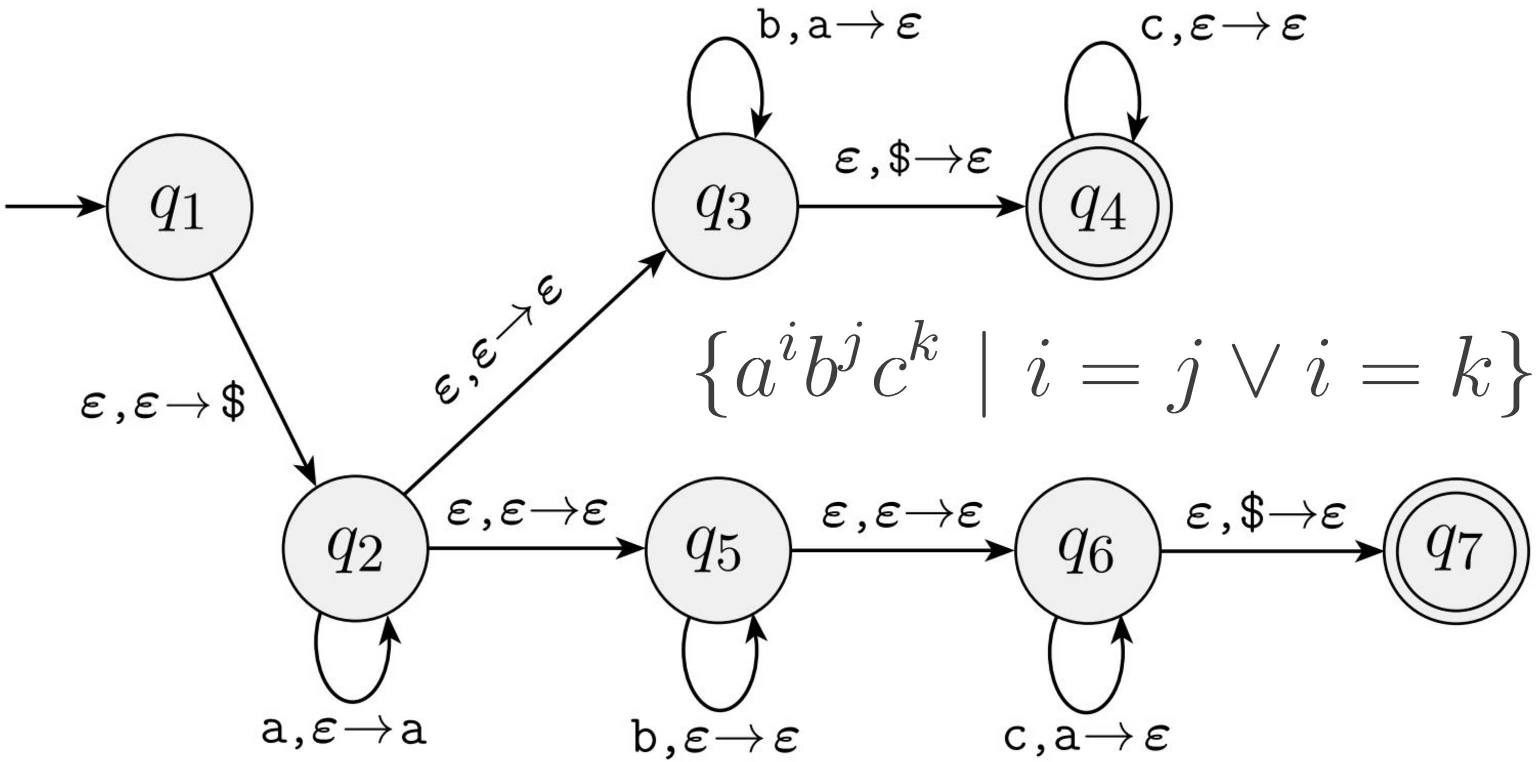

$$\{a^i b^j c^k \mid i = j \vee i = k\}$$