Context Free Grammar $G = (V, \Sigma, R, S)$ where

- V is Variables, a finite set of symbols.
- $\Sigma$ is Terminals, a finite set of symbols, disjoint from V.
- R is set of rules / transformations.
- S is the start variable.

Example : $G = (\{S\}, \{0\}, R, S)$, where $R = \{S \rightarrow 0S, S \rightarrow 0\}$
Sample Derivation of 00 : $S \Rightarrow 0S \Rightarrow 00$

Context Free Language (CFL) generated by G.
$L(G) = \{S \Rightarrow^* w \mid w \in \Sigma^*\}$

A language is context free
⇔ Some non-deterministic PDA recognizes it
⇔ It is generated by some CFG

Practice Questions

Question 2.2 Part a : Prove context free languages are not closed under intersection.
Let $A = \{a^m b^n c^n \mid m, n \geq 0\}$
CFG $G_A = (\{S, X, Y\}, \{a, b, c\}, R_A, S)$, where
$R_A = \{S \rightarrow XY, X \rightarrow aX \mid \varepsilon , Y \rightarrow bYc \mid \varepsilon\}$
A is generated by $G_A$. It is a context free language.

Let $B = \{a^n b^n c^m \mid m, n \geq 0\}$
CFG $G_B = (\{S, X, Y\}, \{a, b, c\}, R_B, S)$, where
$R_B = \{S \rightarrow XY, X \rightarrow aXb \mid \varepsilon , Y \rightarrow Yc \mid \varepsilon\}$
B is generated by $G_B$. It is a context free language.

$A \cap B = \{a^n b^n c^n \mid n \geq 0\}$
Example 2.36 in the book proves that $\{a^n b^n c^n \mid n \geq 0\}$ is not context free.
Thus, context free languages are not closed under intersection.

Question 2.2 Part b : Prove context free languages are not closed under complementation.

Assume CFL are closed under complementation.
Let $A = \{a^m b^n c^n \mid m, n \geq 0\}$
Let $B = \{a^n b^n c^m \mid m, n \geq 0\}$

A, B are CFLs
⇒ $A^C$, $B^C$ are CFLs.
⇒ $A^C \cup B^C$ are CFLs // CFLs are closed under union
⇒ $(A^C \cup B^C)^C$ are CFLs.

$(A^C \cup B^C)^C = A \cap B$ is not a CFL. // De Morgan's Law
⇒ Contradiction
⇒ Context free languages are not closed under complementation

Question 2.4 : Give context-free grammars that generate the following languages. $\Sigma = \{0, 1\}$

Part b : {w | w starts and ends with some symbol}
$G = (\{S, T\}, \Sigma, R, S)$, where $R = \{S \rightarrow 0T0 \mid 1T1 \mid \varepsilon,\ T \rightarrow 0T \mid 1T \mid \varepsilon\}$

Part c : {w | w has odd length}
$G = (\{S, T\}, \Sigma, R, S)$, where $R = \{S \rightarrow 0T \mid 1T\ ,\ T \rightarrow 0S \mid 1S \mid \varepsilon\}$

Part e : L = {w | w = reversed w} = {w | w is a palindrome}
$G = (\{S, T\}, \Sigma, R, S)$, where $R = \{S \rightarrow 0S0 \mid 1S1 \mid 1 \mid 0 \mid \varepsilon\}$

Part f : Empty set
$G = (\{S, T\}, \Sigma, R, S)$, where $R = \{\}$